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SOLUTION OF ONE PROBLEM OF FRACTURE MECHANICS

M. V. Gavrilkina, V. V. Glagolev, and A. A. Markin

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This paper considers a model for the opening-mode fracture separation process based on the introduction of an interaction layer. This layer is defined as the region of localization of the fracture process. The stress–strain state of the layer material is uniform in the cross section of the layer. A study is made of the deformation of a double-cantilever beam weakened by a notch whose width is equal to the thickness of the interaction layer. The problem is solved in a linearly geometrical approximation. The thickness of the interaction layer is estimated, and a method for solving the formulated problem is proposed.

Key words: *characteristic size, ideally elastoplastic model, specific work of fracture.*

1. Formulation of the Problem. Experimental studies of the fracture of solids with recesses of various curvature radii have shown that beginning with a certain limiting value of the radius, the fracture load does not depend on this parameter. This suggests the existence of a certain characteristic size for which there is a localization of the fracture process. In [1, 2], this size is defined as the thickness of the interaction layer, which is a characteristic of the material.

We consider an opening-mode fracture process, which in this case is the propagation of a notch whose width is equal to the thickness of the layer δ_0 (Fig. 1). We place the origin of a stationary Cartesian coordinate system at the tip of the notch. The notch faces are loaded by a point force $\boldsymbol{P}$ — the resultant load distributed along the cantilever and applied at a distance a from the coordinate origin. Because accounting for the weakening stage has an insignificant effect on the value of the wedging force [1], we assume that at the moment of formation of new surfaces, the material of the interaction layer $O'K'K''O''$ deforms steadily in the sense of Drucker. Outside the interaction layer, the medium is considered elastic and the displacements of the points K' and K'' are considered equal to zero.

It is required to determine the critical value of the wedging force P_* that corresponds to the beginning of formation of new surfaces, the length of the plastic zone of the interaction layer l_p, and the stress–strain diagrams taking into account the interaction between the edges of the layer and cantilevers in the critical state.

By virtue of the symmetry of the problem, we consider only the upper cantilever ($x_1 \geqslant \delta_0/2$) and replace its interaction with the layer by the sought load $\boldsymbol{q}(x_2)$.

The tangential component of the load exerted on the beam by the interaction layer will be ignored. Thus, the load exerted by the layer on the beam is distributed according to the diagram shown in Fig. 2.

We confine ourselves to the case of small deformations. In this case, the external-load vector on the cantilever segment adjoining the interaction layer is represented as

$$\boldsymbol{q}(x_2) = \boldsymbol{n} \cdot S, \tag{1}$$

where S is the true-stress tensor and $\boldsymbol{n} = -\boldsymbol{e}_1$ is the outward normal vector to the cantilever surface. Taking into account the condition of stress uniformity along the coordinate x_2 in the interaction layer, from (1) we obtain

$$\boldsymbol{q}(x_2) = -S_1 \boldsymbol{e}_1.$$

Tula State University, Tula 300600; awakeningmaria@mail.ru; vadim@tsu.tula.ru; markin@uic.tula.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 4, pp. 121–127, July–August, 2007. Original article submitted June 30, 2006.

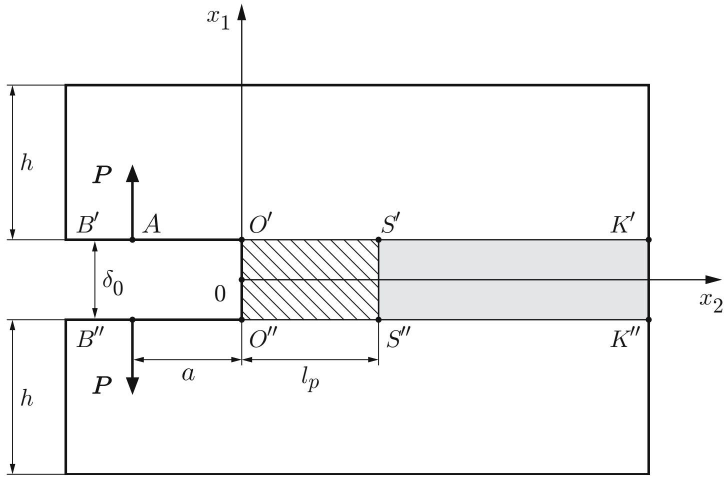

Fig. 1. Loading diagram.

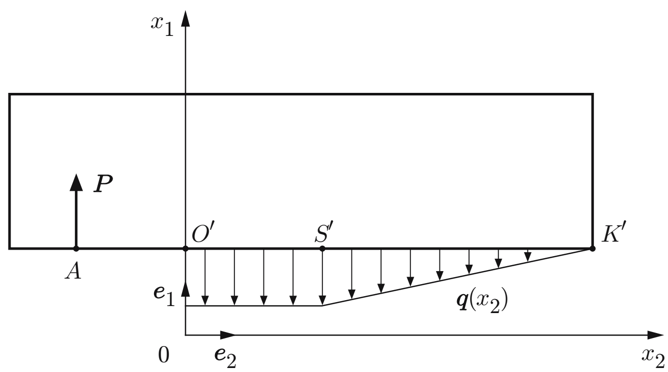

Fig. 2. Load profile.

The material of the interaction layer is considered ideally elastoplastic [3]. Because of the smallness of the deformations and the uniformity of the deformed state of the interaction layer, the linear strain tensor component is expressed as

$$\varepsilon_1 = 2u_1(x_2)/\delta_0, \tag{2}$$

where $u_1(x_2)$ is the displacement of the boundary of the interaction layer ($x_1 = \delta_0/2$) in the x_1 direction. Below, we use the notation $x \equiv x_2$ and $u_1(x_2) = u(x)$.

Relations (1) and (2) imply the following relationship between the projection of the external load onto the x_1 axis and the displacement component:

$$q(x) = \begin{cases} -(2E/\delta_0)u(x), & S' \leqslant x \leqslant K', \\ -S_k, & O' \leqslant x \leqslant S'. \end{cases} \tag{3}$$

Here E is Young modulus of the material, S_k is the yield limit, and $|O'S'| = l_p$ is the length of the plastic deformation region.

The process is described using the beam approximation [4]. The behavior of the cantilever outside the interaction layer is described using the Kirchhoff–Love relations. In view of (3), the bending equation has the following form:

$$D\,\frac{d^2u(x)}{dx^2} = P(a + x) \tag{4}$$

on the segment AO',

$$D\,\frac{d^4u(x)}{dx^4} = -S_k \tag{5}$$

on the segment $O'S'$, or

$$D\,\frac{d^4u(x)}{dx^4} = -\frac{2E}{\delta_0}\,u(x) \tag{6}$$

on the segment $S'K'$. Here $D = Eh^3/12$ is the stiffness of a strip of unit thickness and E is Young modulus of the cantilever material.

Integrating Eqs. (4)–(6) subject to the condition of displacement attenuation at the point K', we obtain the following expressions for the displacement field:

$$u(x) = P(a+x)^3/(6D) + k_1x + k_2 \tag{7}$$

on the segment AO',

$$u(x) = -S_kx^4/(24D) + C_1x^3 + C_2x^2 + C_3x + C_4 \tag{8}$$

on the segment $O'S'$, and

$$u(x) = \mathrm{e}^{-Rx}(L_1\cos Rx + L_2\sin Rx) \tag{9}$$

on the segment $S'K'$. Here $R = \sqrt[4]{E/(2D\delta_0)}$.

For system (7)–(9), we write the continuity conditions for the displacement $u(x)$, the rotation angle $u'(x)$, the bending moment $Du''(x)$, and the shear force $Du'''(x)$ at the points O' $(x = 0)$ and S' $(x = l_p)$. Once the wedging force reaches the critical value P_*, the displacement at the point O' becomes equal to $u_* = \varepsilon^*\delta_0/2$ and that at the point S' to $u_0 = \varepsilon^0\delta_0/2$ (ε^0 is the strain elastic limit). As a result, we obtain the following system of nonlinear equations for the length of the plastic zone l_p, the critical force P_*, and the integration constants k_1, k_2, L_1, L_2, $C_1,\ldots,C_4$:

$$\begin{gathered}
P_*a^3/(6D) + k_2 = C_4, \qquad P_*a^2/(2D) + k_1 = C_3, \qquad P_*a/D = 2C_2, \qquad P_*/D = 6C_1,\\
-S_kl_p^4/(24D) + C_1l_p^3 + C_2l_p^2 + C_3l_p + C_4 = L_1\,\mathrm{e}^{-Rl_p}\cos Rl_p + L_2\,\mathrm{e}^{-Rl_p}\sin Rl_p,\\
-S_kl_p^3/(6D) + 3C_1l_p^2 + 2C_2l_p + C_3 = R\,\mathrm{e}^{-Rl_p}[-L_1(\cos Rl_p + \sin Rl_p) + L_2(\cos Rl_p - \sin Rl_p)],\\
-S_kl_p^2/(2D) + 6C_1l_p + 2C_2 = 2R^2\,\mathrm{e}^{-Rl_p}[L_1\sin Rl_p - L_2\cos Rl_p],\\
-S_kl_p/(6D) + 6C_1 = 2R^3\,\mathrm{e}^{-Rl_p}[L_1(\cos Rl_p - \sin Rl_p) + L_2(\cos Rl_p + \sin Rl_p)],\\
-S_kl_p^4/(24D) + C_1l_p^3 + C_2l_p^2 + C_3l_p + C_4 = \varepsilon^0\delta_0/2,\\
P_*a^3/(6D) + k_2 = \varepsilon^*\delta_0/2.
\end{gathered} \tag{10}$$

Thus, finding the basic fracture characteristics of the elastoplastic material reduces to solving a system of nonlinear equations for the specified properties of the material.

2. Analysis of the System of Nonlinear Equations and a Method for Its Solution. For a specified thickness of the layer, system (10) is nonlinear for l_p. We consider a version of its solution.

For a fixed value of l_p, system (10) is linear for the unknowns P_*, k_1, k_2, L_1, L_2, and $C_1,\ldots,C_4$ but it is overdetermined. We eliminate the last equation from (10),

$$P_*a^3/(6D) + k_2 = \varepsilon^*\delta_0/2 \tag{11}$$

and use the resulting system of linear equations to determine the values of P_*, k_1, k_2, L_1, L_2, and $C_1,\ldots,C_4$ as functions of l_p. To find the length of the plastic zone of the interaction layer with specified accuracy in solving Eq. (11), we used the bisection method.

To solve system (10), we used the Davidenko method and successively employed a minimization method and the Steffensen method [5]. In the Davidenko method, the rate of convergence is lower than that in the successive use of the minimization (gradient descent) method and the Steffensen method.

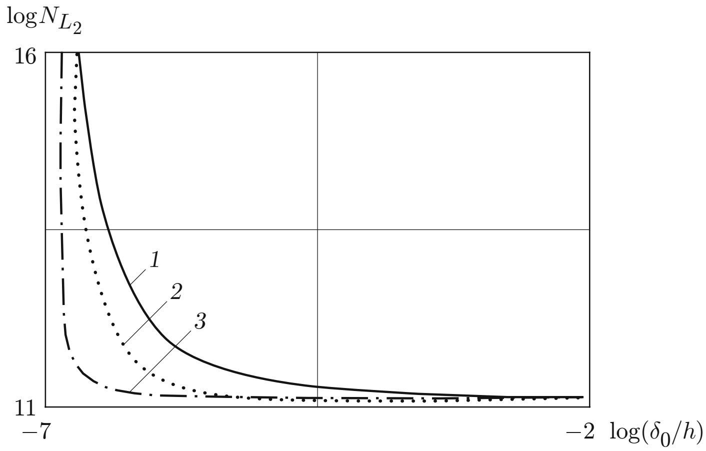

Fig. 3. Conditioning number of the linear system of equations versus thickness of the interaction layer: 1) $l_p = \delta_0$; 2) $l_p = 10\delta_0$; 3) $l_p = 50\delta_0$.

3. Basic Results of the Solution. Let us determine the effect of the cantilever height on the length of the plastic zone l_p.

The initial approximation is found using the system of linear equations for P_*, k_1, k_2, L_1, L_2, and $C_1, \ldots, C_4$. Figure 3 gives the dependence of the conditioning number N_{L_2} [5] of the coefficient matrix (for the norm L_2) of this system on the thickness of the interaction layer for $h = 0.1$ m, $a = 20h$, and the following material characteristics: $E = 2.1 \cdot 10^5$ MPa and $S_k = 600$ MPa. From Fig. 3 it follows that the parameter δ_0 has a significant effect on the conditioning of the system and is a determining factor in choosing a method for solving the problem posed. Similar results for the conditioning number are obtained using the Euclidean and uniform norms of the matrix L_1.

The thickness of the interaction layer can be found using experimental results on the fracture of a double-cantilever beam (DCB) and the solution of system (10) to process them. In this case, it is necessary to experimentally determine the values of P_* and l_p that correspond to the beginning of formation of new material surfaces (the characteristics ε^0 and ε^* are considered known). Since there are no experimental data, we give a number of estimates of the layer thickness in terms of well-known mechanical characteristics. In [2], the layer thickness was expressed as $\delta_0 \approx 0.01 b_0/(\varepsilon^0)^2$, where b_0 is the interatomic spacing. Assuming that for the majority of metals, the value of b_0 is on the order of 10^{-10}–10^{-9} m and $\varepsilon^0 = 10^{-3}$, we obtain $\delta_0 \approx 10^{-6}$–10^{-5} m.

The next estimate can be obtained if the fracture toughness $K_{\rm Ic}$ is known. In this case, we consider the expression for the work of the external forces expended in increasing the fracture surface of infinitesimal area α in the case of the evolution of the mathematical and physical notches in the model proposed under active loading by an external load $\boldsymbol{P}$ (see Fig. 1). In the model with a mathematical notch, the corresponding work of the external forces will be denoted by A_k^m, and in the model with the interaction layer, it will be denoted by A_k^s. In addition, the stresses acting on the surfaces formed will be considered external stresses. The works of these stresses will be denoted by A_m and A_s. Thus, the work of the external forces can be expressed as

$$A_k^m + A_m = A_k^s + A_s. \qquad (12)$$

Since for quasibrittle fracture, the work of the external load $\boldsymbol{P}$ does not depend on the choice of the model and is a constant [6], we have $A_k^m = A_k^s$ and, hence, from (12) we find that $A_m = A_s$. The last equality is also valid for the specific works in the case of an infinitesimal increment in the fracture surface:

$$A'_m = A'_s. \qquad (13)$$

Here $A'_m = \lim\limits_{\alpha \to 0} A_m/\alpha$ and $A'_s = \lim\limits_{\alpha \to 0} A_s/\alpha$.

For the model with a mathematical notch within the framework of the asymptotic solutions of linear elasticity theory, we obtain the well-known expression [6, 7]

$$A'_m = \lim_{\alpha \to 0} \frac{1}{\alpha} \int\limits_0^{\alpha} S_1(x) u(x)\, dx = \frac{K_{\rm Ic}^2}{E}. \qquad (14)$$

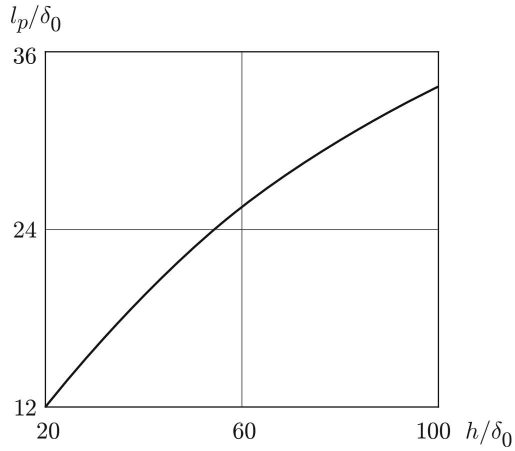

Fig. 4. Length of the plastic zone versus cantilever height for $\delta_0 = 10^{-5}$ m.

The stress work A_s (per unit thickness of the sample) in a material volume $\alpha\delta_0$ of the interaction layer can be found by the formula

$$A_s = \int_0^{\alpha} \int_0^{\varepsilon(x)} \delta_0 S_k \, d\varepsilon \, dx, \tag{15}$$

where $\varepsilon(x)$ is the strain of the layer in the critical state.

By virtue of the ideally elastoplastic model for the behavior of the material of the interaction layer, expression (15) can be written as

$$A_s = \int_0^{\alpha} \Big(\frac{1}{2} S_k \int_0^{\varepsilon^0} \delta_0 \, d\varepsilon + S_k \int_{\varepsilon^0}^{\varepsilon(x)} \delta_0 \, d\varepsilon\Big) \, dx. \tag{16}$$

From (16) we obtain

$$A_s = \int_0^{\alpha} S_k \delta_0 \varepsilon(x) \, dx - \frac{1}{2} S_k \varepsilon^0 \delta_0 \alpha.$$

Because of the uniformity of the stress–strain state in the interaction layer, the strain is given by (2). Hence, in view of (2), the work per unit length of the surface formed can be found by the formula

$$A_s' = \lim_{\alpha \to 0} \frac{2}{\alpha} \int_0^{\alpha} S_k u(x) \, dx - \frac{1}{2} S_k \varepsilon^0 \delta_0, \tag{17}$$

where $u(x) = -S_k x^4/(24D) + C_1 x^3 + C_2 x^2 + C_3 x + C_4$ by virtue of solution (7)–(9).

Thus, from (17), we obtain the expression

$$A_s' = 2 S_k C_4 - S_k \varepsilon^0 \delta_0 / 2. \tag{18}$$

The integration constant C_4 is determined from the condition that the displacement at the point O' (see Fig. 2) reaches the critical value $u_* = \varepsilon^* \delta_0 / 2$:

$$C_4 = u_* = \varepsilon^* \delta_0 / 2. \tag{19}$$

From the equalities (13), (14), (18), and (19), the thickness of the interaction layer is expressed as

$$\delta_0 = \frac{K_{\mathrm{I}c}^2}{S_k E (\varepsilon^* - \varepsilon^0/2)}. \tag{20}$$

Using the relationship between the fracture toughness and the critical force in the DCB model $P_* = h^{3/2}K_{\rm Ic}/(2\sqrt{3}\,a)$ [7], from (20) we obtain the following estimate for the thickness of the interaction layer in terms of the measured critical force:

$$\delta_0 = \frac{12a^2P_*^2}{h^3S_kE(\varepsilon^* - \varepsilon^0/2)}.$$

According to (20), for materials with pronounced plastic properties ($\varepsilon^* \gg \varepsilon^0$), we have

$$K_{\rm Ic} = \sqrt{S_kE\delta_*}\,, \tag{21}$$

where $\delta_* = \varepsilon^*\delta_0$ is the critical displacement of the interaction layer.

Expression (21) coincides with the expression for fracture toughness in the case of using the Leonov–Panasyuk–Dugdale criterion [7] if δ_* is associated with the critical crack opening and S_k with the stress due to interaction between the crack faces. In [8], the following characteristics for St. 3 steel are given: $K_{\rm Ic} = 81$ MPa $\cdot$ m$^{1/2}$, $\varepsilon^* = 0.33$, $E = 2.1 \cdot 10^5$ MPa, and $S_k = 900$ MPa. Thus, from (21) we obtain an estimate $\delta_0 \approx 10^{-4}$ m, and, hence, the thickness of the interaction layer is in the range 10^{-6}–10^{-4} m.

Figure 4 gives a calculated curve of the length of the plastic zone versus the height of a cantilever made of St. 3 steel for $\delta_0 = 10^{-5}$ m. It is evident that the length of the plastic zone depends greatly on the geometry of the sample and is not a characteristic of the fracture process (which is noted in [9]).

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